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G. Chiavassa
J. Liandrat

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National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-0001

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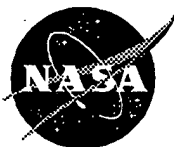IRPHE
12 avenue général Leclerc
13003 Marseille, France

* and ESM2, Université d'Aix marseille II, IMT
Technopole de Chateau Gombert
13451 Marseille Cedex 20, France

ABSTRACT

We construct compactly supported wavelet bases satisfying homogeneous boundary conditions on the interval $[0,1]$. The maximum features of multiresolution analysis on the line are retained, including polynomial approximation and tree algorithms. The case of $H_0^1([0,1])$ is detailed, and numerical values, required for the implementation, are provided for the Neumann and Dirichlet boundary conditions.

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INTRODUCTION

Wavelet bases are often presented as a powerful tool to perform the approximation and the numerical resolution of partial differential equations. Indeed, thanks to zero moment and localization properties, wavelet spaces are self-adapted to the solution and therefore may allow fast and accurate resolution. In the last few years different algorithms have been successfully tested on linear and non-linear equations [9] [12]. Nevertheless, most of the addressed problems where posed in the periodic framework which circumvents the difficulties generated by general boundary conditions, but which makes nearly impossible the treatments of real problems.

It is known [3] that under very general hypotheses, one can only consider homogeneous conditions. Therefore, we are driven to the construction of approximation spaces for homogeneous functional spaces on the interval. For example, for problem involving homogeneous Dirichlet conditions on $[0,1]$, i.e $u(0) = u(1) = 0$, the solution u could be reached in the Sobolev space $H_0^1([0,1]) = \{u \in H^1([0,1]), u(0) = u(1) = 0\}$ and this leads to the construction of approximation spaces for $H_0^1([0,1])$. Various constructions have been proposed (see Auscher [2], in [4] and references in it) but are not, according to their authors, numerically tractable. The proposed construction follows the one of Cohen et al. ([1]). After a short recall of this preliminary construction we introduce and analyze the homogeneous space construction. Numerical details are provided.

I THE PRELIMINARY CONSTRUCTION OF COHEN, DAUBECHIES AND VIAL

This construction of wavelets on the interval, ([1]), is derived from the compactly supported wavelet multi resolution analysis on the line introduced by I. Daubechies [5].

In the case of $L^2(\mathbb{R})$ this multi resolution analysis is classically given by a sequence of closed subspaces V_j satisfying:

i) $... \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...L^2(\mathbb{R})$, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$

Moreover, each V_j is spanned by the translations of the dilated version of a fixed function, the *scaling* function ϕ, i.e $V_j = span\{2^{j/2}\phi(2^j. - k), \quad k \in \mathbb{Z}\}$. Here, the family $\{2^{j/2}\phi(2^j. - k), \quad k \in \mathbb{Z}\}$ is orthonormal and ϕ is on the one hand compactly supported and on the other hand such that its Fourier transform $\hat{\phi}$ satisfies the Strang and Fix approximation rules of order N-1 [7],

$$\hat{\phi}^{(n)}(2k\pi) = 0, \quad k \in \mathbb{Z}\backslash\{0\}, \quad n = 0, ..., N - 1. \tag{1}$$

One consequence of (1) is that the family, $\{\phi(. - k), \quad k \in \mathbb{Z}\}$, can reproduce locally the polynomials of degree at most $N - 1$.

The support of ϕ is the interval $[-N + 1, N]$ and the regularity of ϕ is asymptotically $C^{0.2N}$ [5]. Moreover, ϕ is solution of the following *scaling equation*:

$$\phi(x) = \sum_{k=-N+1}^{N} h_k\phi(2x - k). \tag{2}$$

The detail spaces W_j are defined as the orthogonal complements of V_j in V_{j+1}, i.e,

$$W_j = V_{j+1} \cap \left(V_j\right)^{\perp} \tag{3}$$

and, thanks to *i)*

$$\overline{\bigcup_{j \in \mathbb{Z}} W_j} = L^2(\mathbb{R}).$$

The essential feature of multi resolution analysis (see Y. Meyer [11]) is that $\exists \psi$ such that $\forall j \in \mathbb{Z}$

$$W_j = span\{2^{j/2}\psi(2^j. - k), \quad k \in \mathbb{Z}\}.$$

Again, the family $\{2^{j/2}\psi(2^j. - k), \quad k \in \mathbb{Z}\}$ is orthonormal. The function ψ is here a compactly supported wavelet and is obtained from the following *detail equation*

$$\psi(x) = \sum_{k=-N+1}^{N} g_k\phi(2x - k) \tag{4}$$

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Moreover $supp(\psi) = supp(\phi)$ and ψ has the same regularity as ϕ. In addition, because of the approximation properties of V_0 and the definition of W_0, ψ has got N vanishing moments, i.e:

$$\int x^l \psi(x) = 0 \qquad l = 0, ..., N - 1.$$

Finally, the family $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\}$ is an unconditional basis for various functional spaces such as Hölder spaces $C^s(\mathbb{R})$ or Sobolev spaces $H^s(\mathbb{R})^\dagger$.

In [1], the goal of I. Daubechies et al. was to construct a family of wavelet basis on the interval [0,1] able to characterize $L^2([0,1])$, $H^s([0,1])$ or $C^s([0,1])$ while preserving the most attractive properties of multi resolution analysis of $L^2(\mathbb{R})$, despite the lack of shift invariance of $L^2([0,1])$ (this is not the case for the constructions of P. Auscher (in [4])).

We give in the following paragraph an outline of the construction but the reader should refer to [1] for details.

The construction is performed in two steps as follows.

The first step consists in defining suitable subspaces of $L^2([0,1])$ from a basis *essentially* constructed from the translated versions of a rescaled function, while the second step consists in the construction of the detail spaces with the same requirement.

More precisely, in the **first step**, $V_j([0,1])$ is constructed as follows:

Thanks to the compact support of ϕ, for large enough values of j and $k = N, ..., 2^j - N - 1$, the support of the functions $\phi(2^j x - k)$ is included in [0,1]. Therefore, the corresponding functions may be used as the *interior basis functions* of $V_j([0,1])$ and the set $\Phi_I = \{\phi(2^j x - k), \quad k = N, ..., 2^j - N - 1\}$ ($_I$ stands for interior) is then defined. To fully define $V_j([0,1])$, N *edge functions* are added at each boundary of [0,1] to complete the basis Φ_I. These two families of N functions, $\Phi_{E,0} = \{\varphi_{j,k}^0, \quad k = 0, ..., N-1\}$ and $\Phi_{E,1} = \{\varphi_{j,k}^1, \quad k = 2^j - N, ..., 2^j - 1\}$ are constructed to have minimal support and, such that the order of approximation, (N), related to the interior functions is kept. In other words, all polynomials of degree less than $N - 1$ should be locally expandable as a linear combination of the basis functions of $V_j([0,1])$. Let us

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†We remind that , $s \in \mathbb{R}$, f belongs to $H^s(T)$ if and only if $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2(1 + n^2)^s < +\infty$ and that for $0 < \alpha < 1$, $f \in C^\alpha$ if and only if $|f(x + h) - f(x)| \le C|h|^\alpha$ for every x, h in $\mathbb{R}$, the constant C not depending on x and h.

recall the construction of $\Phi_{E,0}$ (the same applies to $\Phi_{E,1}$). The edge functions $\Phi_{E,0}$ are defined for $k = 0, ..., N-1$ as the restriction to [0,1] of a specific linear combination of the family $\{\phi(x-k)$ such that $0 \in supp(\phi(x-k)\}$. More precisely,

$$\varphi_{j,k}^0(x) = 2^{j/2} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j x + n - N + 1) \; \chi_{[0,1]} \tag{5}$$

The supports $supp(\varphi_{j,k}^0)$ are staggered, i.e, $supp(\varphi_{j,k}^0) = [0, \frac{2N-1-k}{2^j}]$ and $\varphi_{j,k}^0$ is a polynomial of degree k on the interval $[0, \frac{1}{2^j}]$ (see Figure 1 for example). Indeed, $\sum_n n^k \phi(x-n)$ is a polynomial of order k [1] and $\binom{n}{k}$ is a polynomial in n of degree k.

By construction, $\Phi_{E,0} \perp \Phi_I$ but $\Phi_{E,0}$ is not an orthonormal family. An orthonormalization procedure using the Gram-Schmidt algorithm is then performed. Starting from $\varphi_{j,N-1}^0$ down to $\varphi_{j,0}^0$ one obtains N orthonormal edge functions $\{\tilde{\varphi}_{j,k}^0, \; k = 0, ..., N-1\}$ with staggered support $[0, N+k]$ and still, $\tilde{\varphi}_{j,k}^0|_{[0,2^{-j}]}$ is a polynomial of degree k.
Finally, $V_j([0,1])$ is then by definition generated by the orthonormal family $\Phi_{E,0} \cup \Phi_I \cup \Phi_{E,1}$:

$$V_j([0,1]) = span \left\{ \begin{array}{c} \{\tilde{\varphi}_{j,k}^0, \quad k = 0, ..., N-1\} \\ \cup \\ \{\phi_{j,k}, \quad k = N, ..., 2^j - N - 1\} \\ \cup \\ \{\tilde{\varphi}_{j,k}^1, \quad k = 2^j - N, ..., 2^j - 1\} \end{array} \right\} \tag{6}$$

with $\phi_{jk} = 2^{j/2} \phi(2^j x - k)$. One gets

$$V_{j_0}([0,1]) \subset V_{j_0+1}([0,1]) \subset ... \subset V_j([0,1]) \subset ... L^2([0,1])$$

where j_0 is chosen so that $support(\Phi_{E,0}) \cap support(\Phi_{E,1}) = \emptyset$. As ϕ_{jk}, the edge functions satisfy a *modified scaling equation* (2), one writes

$$\tilde{\varphi}_{j,k}^0 = \sum_{n=0}^{N-1} h_{k,n}^0 \tilde{\varphi}_{j+1,n}^0 + \sum_{n=N}^{N+2k} h_{k,n}^0 \phi_{j+1,n} \qquad k = 0, ..., N-1 \tag{7}$$

The numerical values of the coefficients $\{h_{k,n}^0, \quad n = 0, ..., N+2k; \quad k = 0, ..., N-1\}$ and $\{h_{k,n}^1, \quad n = 3*2^j - N - 2k - 3, ..., 2^j - 1; \quad k = 2^j - N, ..., 2^j - 1\}$

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for the right edge, are computed in [1].

The **second step** of the construction is the definition of a suitable basis for the usual wavelet space $W_j([0,1]) = V_{j+1}([0,1]) \cap \left(V_j([0,1]) \right)^{\perp}$. Thanks to (3) and to the compact support of ψ, for large values of j , the family $\Psi_I = \{\psi(2^j x - k), \quad k = N, ..., 2^j - N - 1\}$ belong to $W_j([0,1])$. Wavelets of this family constitute a first part of the basis of $W_j([0,1])$ and are called the interior wavelets. Since $dim(W_j([0,1]) = 2^j$, N other wavelets at each edge should be added to Ψ_I. Again, we only recall the construction at the edge $x = 0$. The complementary wavelets are deduced from the definition of $W_j([0,1])$ as

$$\psi_{jk}^0 = \tilde{\varphi}_{j+1,k}^0 - \sum_{n=0}^{N-1} \langle \tilde{\varphi}_{j+1,k}^0, \tilde{\varphi}_{j,n}^0 \rangle \ \tilde{\varphi}_{j,n}^0 \qquad k = 0, ..., N-1 \qquad (8)$$

where $\langle .,. \rangle$ stands for the scalar product of $L^2([0,1])$. By construction they are orthogonal to $V_j([0,1])$ and to the interior wavelets. Their supports are no longer staggered, but an iterative process described in [1] reduces the support of ψ_{jk}^0 to $[0, \frac{N+k}{2^j}]$ instead of $[0, \frac{2N-1}{2^j}]$. The last step of this construction consists again of a Gram-Schmidt orthonormalization. Starting from $k = 0$ up to $N-1$ one gets N orthonormalized wavelets for the left edge, $\{\tilde{\psi}_{jk}^0, \ k = 0, ..., N-1\}$. These wavelets are known through the coefficients $\{g_{k,n}^0, \quad n = 0, ..., N+2k; \quad k = 0, ..., N-1\}$ that occur in the *modified details equation*:

$$\tilde{\psi}_{j,k}^0 = \sum_{N=0}^{N-1} g_{k,n}^0 \tilde{\varphi}_{j+1,n}^0 + \sum_{n=N}^{N+2k} g_{k,n}^0 \phi_{j+1,n} \qquad k = 0, ..., N-1. \qquad (9)$$

$W_j([0,1])$ is therefore entirely characterized by

$$W_j([0,1]) = span \left\{ \begin{array}{c} \{\tilde{\psi}_{j,k}^0, \ k = 0, ..., N-1\} \\ \bigcup \\ \{\psi_{jk}, \ k = N, ..., 2^j - N - 1\} \\ \bigcup \\ \{\tilde{\psi}_{j,k}^1, \ k = 2^j - N, ..., 2^j - 1\} \end{array} \right\} \qquad (10)$$

Since

$$L^2([0,1]) = V_{j_0}([0,1]) \bigoplus_{j \geq j_0} W_j([0,1])$$

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one gets an orthonormal basis of $L^2([0,1])$ as:

$$\left\{ \begin{array}{c} \{\tilde{\varphi}^0_{j_0,k}, \quad k = 0, ..., N-1\} \\ \bigcup \\ \{\phi_{j_0,k}, \quad k = N, ..., 2^{j_0} - N - 1\} \\ \bigcup \\ \{\tilde{\varphi}^1_{j_0,k}, \quad k = 2^{j_0} - N, ..., 2^{j_0} - 1\} \end{array} \right\} \bigcup_{j \geq j_0} \left\{ \begin{array}{c} \{\tilde{\psi}^0_{j,k}, \quad k = 0, ..., N-1\} \\ \bigcup \\ \{\psi_{jk}, \quad k = N, ..., 2^j - N - 1\} \\ \bigcup \\ \{\tilde{\psi}^1_{j,k}, \quad k = 2^j - N, ..., 2^j - 1\} \end{array} \right\} \tag{11}$$

Remarks:

As we have said before, these wavelets bases are very attractive because they preserve the main features of the whole line construction. More precisely, since the edge functions are finite linear combinations of some shifts of ϕ, they have the same regularity. From their definition the edge scaling functions generate all the polynomials up to degree $N - 1$ which ensure an order N approximation over all the interval, and the existence of N vanishing moments for the edge wavelets. With these oscillations and enough regularity, these wavelets basis form an unconditional basis for the Hölder spaces $C^s([0,1])$ [1]. The fast wavelet transform [10] which is essential for most numerical applications is preserved even near the boundary thanks to the modified scaling, (7) and detail, (9) relations.

Our aim is to adapt this construction to obtain wavelet families generating functional spaces with homogeneous boundary conditions. More precisely we will consider the following constraints

$$f^{(CL0)}(0) = f^{(CL1)}(1) = 0,$$

where $f^{(i)}$ is the i-th derivative of f.

As will be shown, most of the above construction will be preserved as well as numerical efficiency.

II MULTI RESOLUTION ANALYSIS WITH HOMOGENEOUS BOUNDARY CONDITIONS

This section is devoted to the construction and the properties of compactly supported wavelet satisfying homogeneous conditions of type $f^{(CL0)}(0) = f^{(CL1)}(1) = 0$.

The starting point has been described in the previous section and, keeping the same notations, we now assume that the compactly supported wavelets on the line satisfy $0 \leq CL0, CL1 \leq N - 1$ and $r > max(CL0, CL1)$. Therefore, the spaces $V_j([0,1])$ defined in (6) are included in $C^s([0,1])$ with $r > s \geq max(CL0, CL1)$.

II.1 Construction

As in the previous section, we only focus on the left edge $x = 0$.

According to (6), every function $f_j \in V_j([0,1])$ is written

$$f_j(x) = \sum_{k=0}^{N-1} c_{j,k} \tilde{\varphi}_{j,k}^0 + \sum_{k=N}^{2^j-N-1} c_{j,k} \phi_{j,k} + \sum_{k=2^j-N}^{2^j-1} c_{j,k} \tilde{\varphi}_{j,k}^1. \tag{12}$$

Moreover, only the left edge functions $\Phi_{E,0}$ are non zero around $x = 0$ and then

$$f_j^{(CL0)}(0) = \sum_{k=0}^{N-1} c_{j,k} \left(\tilde{\varphi}_{j,k}^0 \right)^{(CL0)}(0). \tag{13}$$

Therefore one way to impose $f^{(CL0)}(0) = 0$ is to enforce that all the left edge scaling functions satisfy this condition. Following P. Auscher (in [2]), this constraint can be related to a polynomial behavior.

Indeed, from the last section we learned that $\tilde{\varphi}_{j,k}^0$ is a polynomial of degree $N - 1$ on the interval $[0, \frac{1}{2^j}]$, say for example $p_{j,k}(x) = a_{j,k}^0 + a_{j,k}^1 x + ... + a_{j,k}^{N-1} x^{N-1}$. The $CL0$-nth derivative of $\tilde{\varphi}_{j,k}^0$ at 0 is then equal to $a_{j,k}^{CL0}$. Therefore $\left(\tilde{\varphi}_{j,k}^0 \right)^{(CL0)}(0) = 0 \iff a_{j,k}^{CL0} = 0$. The construction of scaling functions satisfying $\varphi^{(CL0)}(0) = 0$ is then equivalent to the construction of edge functions *such that their restrictions to [0,2^{-j}] as no component on the*

monomial x^{CL0}.

The **first step** of our algorithm is then to construct a family of N edge scaling functions $\Phi_{E,0} = \{\tilde{\varphi}_{j,k}^{0,\perp}, \quad k = 0, ..., N-1\}$ and one of N edge wavelets $\Psi_{E,0} = \{\tilde{\psi}_{j,k}^{0,\perp}, \quad k = 0, ..., N-1\}$ with the particularity that only *one* scaling function and *one* wavelet contain x^{CL0} on their polynomial part. The **second step** is to remove the scaling function containing x^{CL0} and to modify the corresponding wavelets. For simplicity we work on the interval $[0,+\infty[$ with a zero dilation scale $(j = 0)$, omitted in the next notations. Moreover, we call $p_k(x) = a_k^0 + a_k^1 x + ... + a_k^k x^k$ the restriction of $\varphi_k^0(x)$ on $[0,1]$.

We start with the first N edge scaling functions (5) of section I. They are defined with the coefficients $\alpha_{k,n}^0$ so that

$$\varphi_k^0(x) = \sum_{n=0}^{k} \alpha_{k,n}^0 \varphi_n^0(2x) + \sum_{n=N}^{3N-2k-2} \alpha_{k,n}^0 \phi(2x - n) \qquad k = 0, ..., N-1 \,(14)$$

(see [1] for the computation of these coefficients). The following proposition tells us how to modify φ_k^0 to eliminate x^{CL0} in the polynomials $p_k, k \neq CL0$. We call $\tilde{\varphi}_k^0$ the new functions and $\tilde{p}_k(x) = \sum_{i=0}^{i=k} \tilde{a}_k^i x^i$

Proposition II.1 *The family $\{\tilde{\varphi}_k^0, \quad k = 0, ..., N-1\}$ defined by:*

$$\begin{cases} \tilde{\varphi}_k^0 = \varphi_k^0 & k = 0, ..., CL0 \\ \tilde{\varphi}_k^0 = \varphi_k^0 - \lambda_k \varphi_{CL0}^0 & k = CL0+1, ..., N-1 \end{cases} \tag{15}$$

with

$$\lambda_k = \frac{\alpha_{CL0,k}^0 + \sum_{i=CL0+1}^{k-1} \alpha_{i,k}^0 \lambda_i}{\alpha_{CL0,CL0}^0 - \alpha_{k,k}^0}$$

is such $a_k^{CL0} = 0, \forall k \neq CL0$.

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Proof:

The existence of λ_k is always ensured for $k \neq CL0$ since $\alpha_{k,k}^0 = 2^{-k}$ [1]. Because p_k is a polynomial of degree k, there is nothing to change for $k \leq CL0$, and therefore $\tilde{\varphi}_k^0 = \varphi_k^0$ as well as $\tilde{p}_k(x) = p_k(x)$ for $k \leq CL0$. Given $k > CL0$, let us suppose that $\forall l < k, l \neq CL0, a_k^{CL0} = 0$. From relation (14) we obtain the following scaling relation for $\tilde{\varphi}_k^0$:

$$\tilde{\varphi}_k^0(x) = \sum_{i=0}^{i=CL0-1}(\alpha_{k,i}^0 - \lambda_k \alpha_{CL0,i}^0) \ \tilde{\varphi}_i^0(2x)$$

$$+\mathbf{A_{k,CL0}} \ \tilde{\varphi}_{CL0}^0(2x) + \alpha_{k,CL0+1}^0 \ \tilde{\varphi}_{CL0+1}^0(2x) + ... + \alpha_{k,k}^0 \ \tilde{\varphi}_k^0(2x)$$

$$+\sum_{n=N}^{3N-2k-2}(\alpha_{k,n}^0 - \lambda_k \alpha_{CL0,n}^0) \ \phi(2x - n)$$

(16)

where

$$\mathbf{A_{k,CL0}} = \alpha_{k,CL0}^0 - \lambda_k \ (\alpha_{CL0,CL0}^0 - \alpha_{k,k}^0) + \sum_{i=CL0+1}^{k-1} \alpha_{i,k}^0 \lambda_i.$$

Since $\phi(2x - n)|_{[0,1/2]} \equiv 0$ for $n \geq N$, the contribution of the third RHS term of 16 to $\tilde{p}_k^0$ is 0. Moreover, for $0 \leq i < k, \tilde{\varphi}_i^0(2x) = \tilde{p}_i^0(2x)$ are polynomial with no component on x^{CL0}. Therefore, the contribution of $\tilde{p}_k^0$ to x^{CL0} is entirely due to $A_{k,CL0}$ and $A_{k,CL0} = 0$ is the condition we are looking for, that completes the proof. ∎

For $k \geq CL0$ the supports of $\tilde{\varphi}_k^0$ are no longer staggered but, in compensation $\forall k, 0 \leq k \leq N - 1, \tilde{\varphi}_k^0|_{[0,1]}$ is still a polynomial of degree k. Therefore the functions $\tilde{\varphi}_k^0, 0 \leq k \leq N - 1$ are independent. Moreover, they are orthonormal to the $\{\phi(x - n), \ n \geq N\}$ since they are linear combinations of the $\{\varphi_k^0, \ k = 0, ..., N - 1\}$.

Following the previous section we now orthogonalize the family $\{\tilde{\varphi}_k^0, 0 \leq k \leq N - 1\}$, keeping the "monomial independence." The only thing to do is: *to exchange the place of $\tilde{\varphi}_{CL0}^0$ and $\tilde{\varphi}_{N-1}^0$ before starting the algorithm from index 0 up to N-1.* The result is an orthonormal family of N edge scaling functions $\{\tilde{\varphi}_k^{0,\perp}, \ k = 0, ..., N - 1\}$ with the particularity that only $p_{N-1}^\perp$, the restriction of $\tilde{\varphi}_{N-1}^{0,\perp}$ on [0,1], contains x^{CL0}. They satisfy a *modified scaling equation:*

$$\tilde{\varphi}_k^{0,\perp}(x) = \sum_{n=0}^{N-1} H_{k,n}^0 \tilde{\varphi}_n^{0,\perp}(2x) + \sum_{n=N}^{3N-2} H_{k,n}^0 \phi(2x - n) \qquad k = 0, ..., N - 1 \text{(17)}$$

with $H^0_{k,N-1} = 0$ for $k = 0, ..., N-2$.

The construction of the N edge scaling functions for the right edge comes from the same algorithm for the half line]-∞,0]. The $\{\tilde{\varphi}^{1,\perp}_k, \quad k = 2^j - N + 1, ..., 2^j - 1\}$ are independent of x^{CL1} and only $\tilde{\varphi}^{1,\perp}_{2^j-N}$ contains this monomial on [0,1].

After a dilatation of 2^j for the 0 and 1 edge functions and adding the $2^j - 2N$ interior scaling functions $\phi_{j,k}$, one gets therefore a new orthonormal basis of $V_j([0,1])$, the space defined by (6). In this family, only $\tilde{\varphi}^{0,\perp}_{j,CL0}$ (resp. $\tilde{\varphi}^{1,\perp}_{j,CL1}$) contributes to x^{CL0} (resp. x^{CL1}) on $[0, 2^{-j}]$ (resp. $[1 - 2^{-j}, 1]$).

To perform our first step of construction we now continue by isolating a single wavelet containing x^{CL0} on $[0, 1/2]$.

As in the previous section N wavelets at each boundary should be added to the interior family Ψ_I. Focusing again on the left edge, we construct a first family following (8) as

$$\psi^0_k(x) = \tilde{\varphi}^{0,\perp}_k(2x) - \sum_{n=0}^{N-1} \langle \tilde{\varphi}^{0,\perp}_k(2x), \tilde{\varphi}^{0,\perp}_n(x) \rangle \; \tilde{\varphi}^{0,\perp}_n(x) \qquad k = 0, ..., N-1 \quad (18)$$

Again, each function $\psi^0_k(x)$ is *polynomial* on the interval [0,1/2].

However, since for all k, ψ^0_k depends on $\tilde{\varphi}^{0,\perp}_{N-1}$, all the ψ^0_k contains the monomial x^{CL0} and are therefore not suitable for our first step (we remind that we want to construct a family of edge wavelets such that only one contains x^{CL0} on $[0, 1/2]$). Still, from (17) and (18) we deduce a modified detail equation for these functions that writes

$$\psi^0_k(x) = \sum_{n=0}^{N-1} \beta^0_{k,n} \tilde{\varphi}^{0,\perp}_n(2x) + \sum_{n=N}^{3N-2} \beta^0_{k,n} \phi(2x - n) \qquad k = 0, ..., N-1. \quad (19)$$

The following proposition tell us how to transform the functions ψ^0_k to reach our first step.

Proposition II.2 *The family* $\Psi_{E,0} = \{\tilde{\psi}^0_k, \quad k = 0, ..., N-1\}$ *given by:*

$$\begin{cases} \tilde{\psi}^0_k = \psi^0_k - \mu_k \psi^0_{N-1} & k = 0, ..., N-2 \\ \tilde{\psi}^0_{N-1} = \psi^0_{N-1} \end{cases} \quad with \quad \mu_k = \frac{\beta^0_{k,N-1}}{\beta^0_{N-1,N-1}} \quad (20)$$

is such that only the restriction of $\tilde{\psi}^0_{N-1}$ *to [0,1/2] contains* x^{CL0}.

Proof: In (19) the monomial x^{CL0} is present only in $\tilde{\varphi}_{N-1}^{0,\perp}$. Writing the details equation for $\tilde{\psi}_k^0$ and canceling the coefficient of $\tilde{\varphi}_{N-1}^{0,\perp}$ gives the desired result. ∎

As previously, we apply a Gram-Schmidt orthonormalization that preserves the above property. Indeed, starting from $\tilde{\psi}_0^0$ we get an orthonormal wavelets family $\Psi_{E,0} = \{\tilde{\psi}_k^{0,\perp}, \ k = 0, ..., N-1\}$ for which only $\tilde{\psi}_{N-1}^{0,\perp}$ contains x^{CL0} on [0,1/2]. These wavelets are defined using the details equation

$$\tilde{\psi}_k^{0,\perp}(x) = \sum_{n=0}^{N-1} G_{k,n}^0 \tilde{\varphi}_n^{0,\perp}(2x) + \sum_{n=N}^{3N-2} G_{k,n}^0 \phi(2x - n) \qquad k = 0, ..., N - 1 \quad (21)$$

It only remains to make this construction again for the right edge with monomial x^{CL1} and to expand all the boundary wavelets of a factor 2^j. Together with interior wavelets family Ψ_I, they form an orthonormal basis of $W_j([0,1])$.

We have now reached our first step since we have constructed a basis of scaling functions for $V_j([0,1])$ and a basis of wavelets for $W_j([0,1])$ such that in each family, only one function has a component on x^{CL0} on [0,1/2] and only one function has a component on x^{CL1} on [1-1/2].

As announced, we now perform the **second step** of our construction by removing the function $\tilde{\varphi}_{j,N-1}^{0,\perp}$ on the left edge and the corresponding ones, $\tilde{\varphi}_{j,2^j-N}^{1,\perp}$, for the right edge.

The last technical point is the modification one should make to the wavelet space. We have the following proposition:

Proposition II.3 *Define the subspace* $\tilde{V}_j([0,1])$ *as*

$$\tilde{V}_j([0,1]) = V_j([0,1]) - span\{\tilde{\varphi}_{j,N-1}^{0,\perp}; \tilde{\varphi}_{j,2^j-N}^{1,\perp}\} \qquad (22)$$

Replace the two wavelets $\tilde{\psi}_{j,N-1}^{0,\perp}$ *and* $\tilde{\psi}_{j,2^j-N}^{1,\perp}$ *in the families* $\Psi_{E,0}$ *and* $\Psi_{E,1}$ *by*

$$\begin{cases} \Theta_j^0 = a\tilde{\varphi}_{j,N-1}^{0,\perp} + b\tilde{\psi}_{j,N-1}^{0,\perp} \\[2mm] \Theta_j^1 = a'\tilde{\varphi}_{j,2^j-N}^{1,\perp} + b'\tilde{\psi}_{j,2^j-N}^{1,\perp} \end{cases} \qquad (23)$$

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with a and b solutions of

$$\begin{cases} aH^0_{N-1,N-1} + bG^0_{N-1,N-1} = 0 \\ a^2 + b^2 = 1 \end{cases} \tag{24}$$

and a', b' solutions of the same set of equations with coefficients for the right edge.

Then the new family $\Psi_{E,0} \cup \Psi_I \cup \Psi_{E,1}$ is an orthonormal basis of $\tilde{W}_j([0,1])$, the orthogonal complement of $\tilde{V}_j([0,1])$ in $\tilde{V}_{j+1}([0,1])$. Moreover every scaling function of $\tilde{V}_j([0,1])$ and every wavelet of $\tilde{W}_j([0,1])$ satisfies the homogeneous boundary conditions $f^{(CL0)}(0) = f^{(CL1)}(1) = 0$.

Proof: We prove only the result for the left edge.

Let us first recall that $\tilde{\varphi}^{0,\perp}_{j,N-1}$ and $\tilde{\psi}^{0,\perp}_{j,N-1}$ are respectively two basis functions of $V_j([0,1])$ and $W_j([0,1])$, and that $V_j([0,1]) \perp W_j([0,1])$. Θ^0_j is then orthogonal to *all the other basis functions of $\tilde{V}_j([0,1])$ and $\tilde{W}_j([0,1])$.* Moreover, $||\Theta^0_j||^2_{L^2} = 1$ if and only if $a^2 + b^2 = 1$. The same argument holds for Θ^1_j and therefore, with the new definition of $\Psi_{E,0}$ and $\Psi_{E,1}$, $\Psi_{E,0} \cup \Psi_I \cup \Psi_{E,1}$ is a family of 2^j orthonormal functions.

Using the scaling (17) and detail (21) equations we get

$$\Theta^0_j = \sum_{n=0}^{N-1} \left(aH^0_{N-1,n} + bG^0_{N-1,n} \right) \tilde{\varphi}^{0,\perp}_{j+1,N} + \sum_{n=N}^{3N-2} \left(aH^0_{N-1,n} + bG^0_{N-1,n} \right) \phi_{j+1,N}.$$

Taking into account (24), we get that that Θ^0_j is independent of $\tilde{\varphi}^{0,\perp}_{j+1,N-1}$ and consequently belongs to $\tilde{V}_{j+1}([0,1])$. Since the orthonormal collection $\Psi_{E,0} \cup \Psi_I \cup \Psi_{E,1}$ generates a closed subspace of $\tilde{V}_{j+1}([0,1])$, orthogonal to $\tilde{V}_j([0,1])$ and of dimension $2^j = dim\tilde{V}_{j+1}([0,1]) - dim\tilde{V}_j([0,1])$, it is by definition $\tilde{W}_j([0,1])$ the orthonormal complement of $\tilde{V}_j([0,1])$ in $\tilde{V}_{j+1}([0,1])$.

■

Remarks:

All these edge functions have the same regularity as the initial scaling function ϕ.

Moreover the $2N - 2$ edges wavelets constructed before we remove $\tilde{\varphi}_{j,N-1}^{0,\perp}$ and $\tilde{\varphi}_{j,2^j-N}^{1,\perp}$, have conserved their N vanishing moments. The modified wavelets Θ_j^0 and Θ_j^1 belong to $\tilde{W}_j([0,1])$ and are therefore orthonormal to all the polynomials included in $\tilde{V}_j([0,1])$. But they have no reason to verify $\langle \Theta_j^0, x^{CL0} \rangle = \langle \Theta_j^1, x^{CL1} \rangle = 0$, since these monomials have been excluded from the edges of $\tilde{V}_j([0,1])$. Hence only one vanishing moment for one wavelet at each boundary has been lost.

As in the initial construction, the modified scaling and detail relations insure that fast algorithms related to the different basis projections are available.

At this point however, we don't know exactly what kind of space the multi resolution family $\tilde{V}_j([0,1])$ approximate. This is the purpose of the next subsection.

II.2   Approximation results

We now check the intuitive result that the wavelet basis derived from the last construction is an orthonormal basis for suitable homogeneous spaces on $[0,1]$. We give a complete proof for the Dirichlet homogeneous boundary conditions $f(0) = f(1) = 0$, i.e, $CL0 = CL1 = 0$, corresponding to $H_0^1([0,1])$.

Using our construction for $CL0 = CL1 = 0$, we first obtain a subspace $V_j([0,1])$ defined by the orthonormal basis $\Phi_{E,0} \cup \Phi_I \cup \Phi_{E,1}$, with the particularity that only two scaling functions are non zero at the edges. It is known that under some specific conditions (see the previous section) $V_j([0,1])$ belongs to a multi resolution analysis of the Sobolev space $H^1([0,1])$.

Let us simplify the notations and write $\tilde{\varphi}_{j,k}^{0,\perp} = \varphi_{j,k}^0$ and $\tilde{\varphi}_{j,k}^{1,\perp} = \varphi_{j,k}^1$. Then we have the following result:

Proposition II.4 *Let $V_j([0,1])$ be the subspace spanned by the orthonormal basis*

$$\{\varphi_{j,0}^0, ..., \varphi_{j,N-1}^0\} \bigcup \{\phi_{j,N}, ..., \phi_{j,2^j-N-1}\} \bigcup \{\varphi_{j,2^j-N}^1, ..., \varphi_{j,2^j-1}^1\}$$

Assume these scaling functions have enough regularity to involve

$$\overline{\bigcup_{j \geq j_0} V_j([0,1])} = H^1([0,1])$$

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and that only $\varphi^0_{j,N-1}$ and $\varphi^1_{j,2^j-N}$ are non zero at 0 and 1. Then

$$\overline{\bigcup_{j \geq j_0} \tilde{V}_j([0,1])} = \overline{\bigcup_{j \geq j_0} (V_j([0,1]) - span\{\varphi^0_{j,N-1}; \varphi^1_{j,2^j-N}\})} = H^1_0([0,1])$$

Proof: Take a function f in $H^1_0([0,1])$, and call $\Pi_j(f)$ and $\tilde{\Pi}_j(f)$ its orthonormal projection onto $V_j([0,1])$ and $\tilde{V}_j([0,1])$. We have to establish the relation

$$\lim_{j \to +\infty} \|f - \tilde{\Pi}_j(f)\|_{H^1} = 0 \tag{25}$$

where the H^1-norm is taken as $\|f\|^2_{H^1} = \|f\|^2_{L^2} + \|\frac{\partial f}{\partial x}\|^2_{L^2}$. Following the density of $V_j([0,1])$ in $H^1([0,1])$ this is equivalent to

$$\lim_{j \to +\infty} \|\Pi_j(f) - \tilde{\Pi}_j(f)\|_{H^1} = 0$$

Now using the orthonormal basis of $V_j([0,1])$ and $\tilde{V}_j([0,1])$ we have

$$\|\Pi_j(f) - \tilde{\Pi}_j(f)\|_{H^1} = |\langle f, \varphi^0_{j,N-1}\rangle| \; \|\varphi^0_{j,N-1}\|_{H^1} + |\langle f, \varphi^1_{j,2^j-N}\rangle| \; \|\varphi^1_{j,2^j-N}\|_{H^1}$$

since the support of the 0 and 1 edges scaling functions do not overlap. Because $\|\varphi^0_{j,N-1}\|_{L^2} = 1$ and $\varphi^0_{j,N-1}$ belongs to $H^1([0,1])$ (due to the regularity of the initial function ϕ), we have

$$\|\Pi_j(f) - \tilde{\Pi}_j(f)\|_{H^1} \leq C_1 2^j \left(|\langle f, \varphi^0_{j,N-1}\rangle| + |\langle f, \varphi^1_{j,2^j-N}\rangle|\right)$$

where C_1 is a constant independent of j. Therefore we have to check that

$$\lim_{j \to +\infty} 2^j |\langle f, \varphi^0_{j,N-1}\rangle| = \lim_{j \to +\infty} 2^j |\langle f, \varphi^1_{j,2^j-N}\rangle| = 0$$

To see this, we use the inequality

$$2^j |\langle f, \varphi^0_{j,N-1}\rangle| \leq C_2 \|f\|_{H^1} \tag{26}$$

with C_2 independent of j, which will be justified at the end of the proof. Let us take now a sequence of functions $(f_n)_{n \in \mathbb{N}}$ convergent to f with respect to

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the H^1-norm, for example $f_n(x) = f(x)\chi_{[\frac{1}{n}, 1-\frac{1}{n}]}$. Applying the last inequality to $f - f_n$ we get

$$2^j |\langle f - f_n, \varphi^0_{j,N-1}\rangle| \leq C_3 \|f - f_n\|_{H^1}.$$

It only remains to note that there exist an integer J, *dependent of n*, for which

$$\forall j \geq J, \quad 2^j |\langle f - f_n, \varphi^0_{j,N-1}\rangle| = 2^j |\langle f, \varphi^0_{j,N-1}\rangle|.$$

Indeed, for fixed n we take J such that the supports of $\varphi^0_{J,N-1}$ and f_n do not overlap.

Making n tends to $+\infty$, and consequently j, leads to

$$\lim_{j \to +\infty} 2^j |\langle f, \varphi^0_{j,N-1}\rangle| = 0.$$

Obviously the same arguments holds for the scalar product $2^j |\langle f, \varphi^0_{j,2^j-N}\rangle|$ and the proposition is proved. ∎

We still have to establish the inequality (26). An integration by parts implies that

$$|\langle f, \varphi^0_{j,N-1}\rangle| \leq \|\frac{\partial f}{\partial x}\|_{L^2} \|\Xi^0_j\|_{L^2}$$

where Ξ^0_j is a primitive of $\varphi^0_{j,N-1}$. Since $\varphi^0_{j,N-1}$ belongs to $L^2([0,1])$ we deduce

$$\|\Xi^0_j\|_{L^2} \leq C_4 2^{-j}.$$

This assumption and the definition of the H^1-norm leads to the desired result. Thanks to this proposition and the definition of $\tilde{W}_j([0,1])$ (see prop II.3) we deduce a decomposition of $H^1_0([0,1])$ in term of wavelet basis,

$$H^1_0([0,1]) = \tilde{V}_{j_0}([0,1]) \bigoplus_{j \geq j_0} \tilde{W}_j([0,1]). \tag{27}$$

Remarks:
The proof for Neumann homogeneous conditions is similar and involves the H^2-norm. More regularity is therefore needed for the basis functions and a double integration by parts to the inequality corresponding to (26). In that case, the approximated space is the strict subspace of $H^2([0,1])$ defined as $\{f \in H^2([0,1]), f^{(1)}(0) = f^{(1)}(1) = 0\}$.

Some mixed homogeneous boundary conditions, for example $f(0) = f(1) = f^{(1)}(0) = f^{(1)}(1) = 0$, could also be addressed with a similar construction. In this case, two scaling functions at each edge are removed from $V_j([0,1])$ and are employed to modify the wavelets of $W_j([0,1])$. This construction leads to a characterization of the functional space
$$H_0^2([0,1]) = \{f \in H^2([0,1]), f(0) = f(1) = 0 \;\; and \;\; f^{(1)}(0) = f^{(1)}(1) = 0\}.$$
Since the left and right basis functions do not interact at scale j, different conditions could also be taken at 0 and 1.

The following section is related to the numerical estimates related to our construction and to various topics connected to its application for partial differential equation problems.

III   NUMERICAL ESTIMATES

This section is devoted to the numerical estimates related to our construction for two cases of homogeneous boundary conditions, i.e, the Dirichlet conditions and the Neumann conditions. All the following computations have been carried out beginning with the initial compactly supported function ϕ closest to linear phase constructed by I. Daubechies [6] with $N = 4$. Since no explicit analytic expressions exist, this function is defined through the *filter* coefficients h_n used in the scaling equation (2). These coefficients are provided in [6]:

$$h_{-3} = -.07576571478950, \quad h_{-2} = -.2963552764600, \quad h_{-1} = .4976186676328$$
$$h_0 = .8037387518051, \quad h_1 = .29785779560531, \quad h_2 = -.0992195435766$$
$$h_3 = -.01260396726203, \quad h_4 = .03222310060405.$$

The corresponding interior wavelet ψ is defined using the coefficients g_n of the details equation (4) with $g_n = (-1)^n h_{2N+1-n}$.

III.1   Dirichlet Boundary conditions

The application of the last section algorithm with $CL0 = CL1 = 0$ (Dirichlet condition) leads to a multi resolution analysis of $H_0^1([0,1])$. Three scaling functions and four wavelets have to be added at each boundary (see section

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II). These scaling functions are solutions of a modified scaling equation (17) and are therefore characterized by the coefficients $H_{k,n}^0$ and $H_{k,n}^1$. The corresponding numerical estimates (computed on a 16 decimal digits computer with an error smaller than 10^{-11}) are listed in Table 1. The coefficients $G_{k,n}^0$ and $G_{k,n}^1$ which occur in the modified detail equation (21) are listed in Table 2 and define completely the edges wavelets. All the following figures are obtained using the cascade algorithm [6]. The three $x = 0$ edge scaling functions, as well as the three $x = 1$ edge scaling functions, are represented on Figure 2 at scale $j = 0$. The corresponding wavelets are plotted on Figure 3. Notice that due to the lack of symmetry of the initial scaling functions and wavelets, the $x = 1$ edge functions can not be deduced from the $x = 0$ edge functions using a simple transformation.

III.2   Neumann Boundary conditions

The same numerical estimates corresponding to the Neumann conditions, i.e CL0=CL1=1, are listed in Tables 3 and 4. The Figures 4 and 5 represent respectively the scaling functions and wavelets of this multi resolution analysis.

Remarks:

Some zero coefficients are provided in Tables 2, 3 and 4. They are expected as follows: for instance in Table 3 $H_{0,1}^0 = H_{0,2}^0 = 0$; since the scaling functions $\varphi_{0,0}^0$ for the Neumann conditions, is by definition constant on the interval $[0,1]$, it does not depend on $\varphi_{1,1}^0$ and $\varphi_{1,2}^0$ which are respectively polynomials of order 1 and 2 on $[0,1/2]$; this leads $H_{0,1}^0 = H_{0,2}^0 = 0$ in Table 3. Others zeros are expected using the same arguments.

III.3   Quadrature formula

In order to use these wavelets basis for numerical purposes one question needs still to be answered. Given a function f, how can we define a projection $V_j([0, 1])$, i.e, how can we estimate a set of coefficients $c_{j,k}$ occuring in relation (12) and corresponding to f? The solution proposed here aims to compute an approximation of the orthogonal projection of f on $V_j([0, 1])$ defining quadrature formula to estimate the coefficients $c_{j,k} = \int f\varphi_{0,k}^0$. We define below a quadrature formula of order $N - 1$ in the same philosophy as G.

Beylkin et al. ([8]) or W. Sweldens et al. ([13]). We are therefore looking for weight coefficients $\omega_{i,k}$ such that

$$\int f \varphi_{0,k}^0 \approx \sum_{i=0}^{N-1} \omega_{i,k} f(a_i) \qquad (28)$$

where the $\{a_i, \ i = 0, ..., N-1\}$ are N given points taken in the support of $\varphi_{0,k}^0$ and such that the approximation is exact for the polynomials of degree less than or equal to $N-1$.

It appears that the weight coefficients $\omega_{i,k}$ are the solution of the following linear system:

$$\int x^l \varphi_{0,k}^0 = \sum_{i=0}^{N-1} \omega_{i,k}(a_i)^l \quad l = 0, ..., N-1 \qquad (29)$$

Hence we need to evaluate the first N moments of every edge scaling function.

Multiplying the modified scaling equation (17) by x^l leads to the $N-1$ equations:

$$\int x^l \varphi_{0,k}^0(x) = \sum_{n=0}^{N-2} H_{k,n}^0 \int x^l \varphi_{0,n}^0(2x) + \sum_{n=N}^{3N-2} H_{k,n}^0 \int x^l \phi(2x-n) \quad k = 0, ..., N-2$$

$$(30)$$

Since the moments of order l of the interior function ϕ, $\int x^l \phi(x)$, can be estimated using the classical recurrence relation given in [8], (30) finally leads to the following linear system $AX_l = b_l$ where the N dimensional vectors X_l and

$$X_l(k) \quad = \quad \int x^l \varphi_{0,k}^0(x)$$

b_l are defined as $\qquad$ and

$$b_l(k) \quad = \quad \sum_{n=N}^{3N-2} H_{k,n}^0 \int x^l \phi(2x-n), k = 0, ..., N-2$$

and where the entries of the matrix A depend only on the $H_{k,n}^0$. We easily checked that this matrix is always nonsingular. (to see this, use the fact that $\sum_{n=0}^{N-1} |H_{k,n}^0| < 1$ since $\|\varphi_{0,k}^0\|_{L^2} = 1$.

We first provide the numerical values of the moments of order $l, 0 \leq l \leq 3$ for $N = 4$:

$$M_0 = 1.00000000000\text{e}+00 \quad M_1 = \text{-}1.45319345240\text{e-}02$$
$$M_2 = 2.11177120898\text{e-}04 \quad M_3 = 4.34510522842\text{e-}02$$

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Then, the entries of X_l for $l = 0, ..., N - 1$ for the 0 and 1 edge scaling functions corresponding to Figure 1 (Dirichlet boundary conditions) are listed in Table 5 and 6.

Using the values of these moments and the N given points a_i, we find the weights $\omega_{i,k}$ for every edge scaling function solving the linear Vandermonde system (29).

Remark:

A quadrature formula of same order has to be used to estimate the interior scaling coefficients $c_{j,k} = \int f \phi_{j,k}, N \leq k \leq 2^j - N - 1$ to preserve a constant order of accuracy all over the interval.

IV   CONCLUSION

Compactly supported wavelets satisfying homogeneous boundary conditions on $[0, 1]$ have been constructed. All the tools required for the use of these functions for numerical approximation of partial differential problems have been detailed.

Even if all this construction extends by tensor product arguments to higher dimensions, efficient handling of general open sets with boundary conditions is still an open problem.

LIST OF SYMBOLS USED

The LaTeX code of the mathematical symbols used is given to clarify their identity:

$H_0^1([0,1])$: H^1_0([0,1]) and a similar code for $L^2(I\!R)$; $H^1([0,1])$; $L^2([0,1])$; $C^s([0,1])$;

$\tilde{V}_j([0,1])$: \tilde{V}_j([0,1]) and same for $\tilde{W}_j([0,1])$; V_j ; W_j ; $V_j([0,1])$; $W_j([0,1])$

$\overline{\bigcup\limits_{j\in Z} V_j}$: \overline{\bigcup_{j \in \Z} V_j} and same for $\bigcap\limits_{j\in Z} V_j$, $\overline{\bigcup\limits_{j\ inZ} W_j}$

$\hat{\phi}^{(n)}(2k\pi)$: {\hat \phi}^{(n)}(2k\pi) ;

$\phi_{j,k}$: \phi_{j,k} ;

$\Phi_{E,0}$: \phi_{E,0} ; Φ_I ; $\Phi_{E,1}$

$\tilde{\varphi}_{j,k}^{0,\perp}$: \tilde{\varphi}^{0,\perp}_{j,k} ; The same expressions are using substituting \varphi by \psi and \Phi by \Psi.

$\binom{n}{k}$: \Bigl(^{n}_{k}\Bigl) ; $\langle,\rangle$: \langle , \rangle ; $\emptyset$: \emptyset

$h_{k,n}^0$: h^0_{k,n} ; $h_{k,n}^1$; $g_{k,n}^0$; $g_{k,n}^1$ and the same expressions with uppercase H and G.

$f^{(CL0)}$: f^{(CL0)} ; $f^{(CL1)}$; $p_{j,k}$:   p_{j,k} ; $\Longleftrightarrow$: \Longleftrightarrow
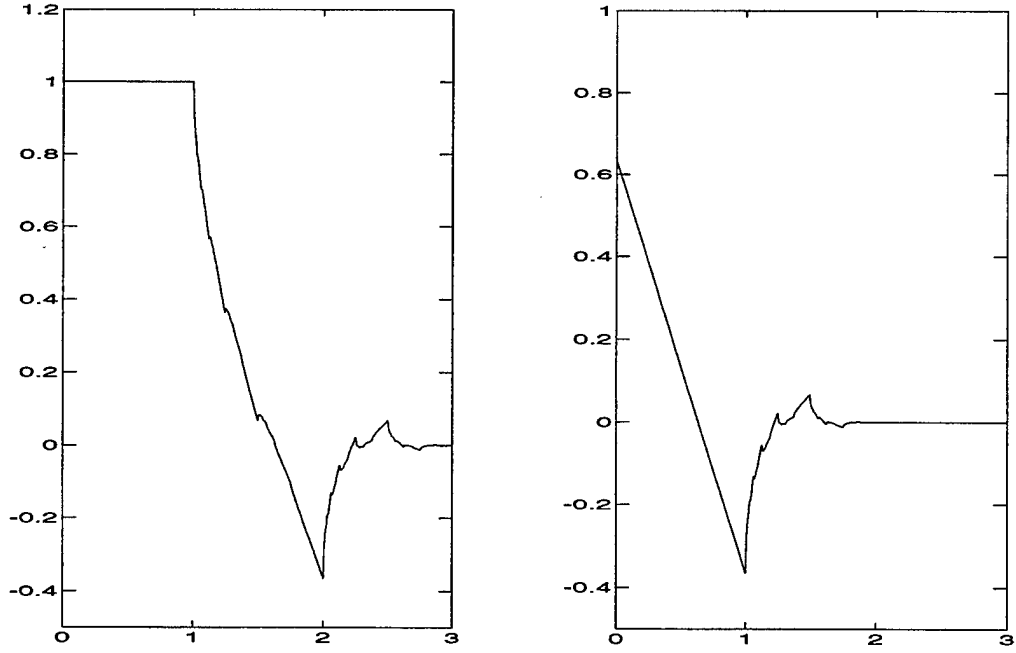
λ_k : \lambda_k ; μ_k : \mu_k ; $\alpha_{k,n}^0$: \alpha^0_{k,n}.

$\beta_{k,n}^0$: \beta^0_{k,n} ; Θ_j^0 : \Theta_j^0 ; $\|.\|_{H^1}$: \|.\|_{ H^1} .

$\Pi_j(f)$: \Pi_j(f) ; $\frac{\partial f}{\partial x}$: \frac{\partial f}{\partial x}

$\chi_{[\frac{1}{n},1-\frac{1}{n}]}$: \chi_{[\frac{1}{n},1-\frac{1}{n}]} ; Ξ_j^0 : \Xi_j^0.

$\approx$: \approx ;

FIG. 1. *The edge scaling functions $\varphi_{0,0}^0$ and $\varphi_{0,1}^0$ for N=2. $\varphi_{0,0}^0$ is constant on [0;1] and $\varphi_{0,1}^0$ is a polynomial of order 1.*

Table 1: **The left and right filter coefficients, $H_{k,n}^0$ and $H_{k,n}^1$, for the construction with Dirichlet homogeneous boundary conditions $f(0) = f(1) = 0$.**

	n	$H_{k,n}^0$	$H_{k,n}^1$
k=0	0	6.3091928199e-01	5.9981737601e-01
	1	9.6514114329e-02	1.0261005527e-01
	2	3.9878718285e-02	2.2276045032e-02
	4	-3.9847422688e-01	-3.8101599048e-01
	5	-4.6171223341e-01	-3.5419483844e-01
	6	-4.3450617898e-01	-3.7772113863e-01
	7	-1.6823324724e-01	-4.0626758997e-01
	8	3.9512280064e-02	-2.2139835764e-01
	9	7.4334018686e-03	1.5553422358e-02
	10	-1.9004116026e-02	3.9763630210e-02
k=1	0	-7.1465107925e-01	-6.7917200335e-01
	1	1.0487856031e-01	7.0564886681e-02
	2	-1.0275008552e-01	-6.1435155066e-02
	4	-2.2668171853e-02	1.0664567825e-01
	5	-3.4790471529e-01	-6.9557725550e-02
	6	-5.4438357904e-01	-3.1266913055e-01
	7	-2.1013565982e-01	-5.5031934226e-01
	8	7.1146281575e-02	-3.3164862537e-01
	9	8.8701254411e-03	2.0464047232e-02
	10	-2.2677220474e-02	5.2318055022e-02
k=2	0	-2.2362773506e-01	-3.4534868269e-01
	1	5.0082651220e-01	3.9554860545e-01
	2	-1.1707940876e-01	-5.1663829154e-02
	4	-7.0092772722e-01	-3.4828534215e-01
	5	-4.8072499606e-02	-5.6934808461e-01
	6	4.1374409814e-01	-1.6719040087e-01
	7	1.2755670687e-01	4.0449494649e-01
	8	-6.4838220288e-02	2.8819962720e-01
	9	-4.9723065158e-03	-1.4195865812e-02
	10	1.2712119110e-02	-3.6292922913e-02

Note: For the right edge ($x = 1$) the coefficients $\{H_{0,n}^1\}$ are related to the scaling function $\varphi_{j,2^j-3}^1$ and are listed from right to left. The case n=3 corresponds to the scaling function we have removed and will therefore not appear.

Table 2: **The left and right wavelet filter coefficients, $G^0_{k,n}$ and $G^1_{k,n}$, for the construction with Dirichlet homogeneous boundary conditions $f(0) = f(1) = 0$.**

	n	$G^0_{k,n}$	$G^1_{k,n}$
k=0	0	2.0137223334e-01	2.4372875232e-01
	1	5.4229707169e-01	4.5801973839e-01
	2	-7.0645626067e-01	-3.4317083287e-01
	4	3.5108313491e-01	7.4312944286e-01
	5	1.5369973568e-01	-1.4784113471e-01
	6	-1.0259167455e-01	-1.8898503160e-01
	7	-7.5615428961e-02	4.5387871214e-02
	8	5.5724919877e-02	3.3997823918e-02
	9	2.6185470422e-03	-1.5613217459e-03
	10	-6.6945353810e-03	-3.9916501406e-03
k=1	0	-6.6848468945e-14	8.0232416876e-14
	1	2.8233692882e-02	3.8745242758e-01
	2	-5.7322775598e-02	-6.7680392343e-01
	4	5.9385536874e-02	-3.9552887470e-01
	5	-1.1474607116e-01	3.7058026047e-01
	6	-2.9055557195e-01	2.8173001140e-01
	7	8.0074010314e-01	-1.0486107295e-01
	8	-4.9704917725e-01	-8.7104529274e-02
	9	-2.9500466849e-02	3.4442984618e-03
	10	7.5420420519e-02	8.8056382200e-03
k=2	0	6.2086522836e-14	-5.0803891077e-13
	1	-1.4802084915e-14	-4.0101204690e-13
	2	6.7504052837e-02	1.0243645353e-02
	4	-3.6141627882e-01	1.1309599230e-01
	5	7.8642373898e-01	-5.4953710005e-01
	6	-4.9310306778e-01	7.6956164180e-01
	7	2.1421713460e-04	-2.8909897939e-01
	8	5.6564958243e-02	-9.0339164667e-02
	9	-1.0869221407e-03	1.2346812450e-02
	10	2.7788077166e-03	3.1565662736e-02
k=3	0	-2.5584258650e-02	-1.6585775990e-02
	1	-6.5876783667e-01	-6.8420143939e-01
	2	-6.8353593454e-01	-6.4585264939e-01
	4	-3.0331012100e-01	2.5993685887e-02
	5	-3.7971289157e-02	-2.7855694418e-01
	6	6.7315440450e-02	-1.5252127585e-01
	7	1.0944670231e-02	8.7162800751e-02
	8	-7.6074757976e-03	7.2638444593e-02
	9	-3.8773215636e-04	-2.8584982261e-03
	10	9.9126981384e-04	-7.3079907303e-03

Note: For the right edge ($x = 1$) the coefficients $\{G^1_{0,n}\}$ are related to the wavelet $\psi^1_{j,2^j-1}$ and are listed from right to left. The case n=3 corresponds to the scaling function we have removed and will therefore not appear.

Table 3: The left and right filter coefficients, $H^0_{k,n}$ and $H^1_{k,n}$, for the construction with Neumann homogeneous boundary conditions $f^{(1)}(0) = f^{(1)}(1) = 0$.

	n	$H^0_{k,n}$	$H^1_{k,n}$
k=0	0	7.0710678119e-01	7.0710678119e-01
	1	0.0000000000e+00	0.0000000000e+00
	2	0.0000000000e+00	0.0000000000e+00
	4	3.7875157560e-01	3.3279641360e-01
	5	4.1933439083e-01	3.1763076592e-01
	6	3.9462541967e-01	3.3872841031e-01
	7	1.5279210977e-01	3.6432796801e-01
	8	-3.5885680933e-02	1.9854306805e-01
	9	-6.7511337557e-03	-1.3947818884e-02
	10	1.7259840309e-02	-3.5658770113e-02
k=1	0	-6.3999878989e-01	-6.2399400714e-01
	1	2.7256158164e-01	2.4746950442e-01
	2	2.8255565264e-02	2.1217268985e-02
	4	9.3033115429e-02	3.1423763181e-02
	5	3.9549596240e-01	1.5138263011e-01
	6	5.4799002074e-01	3.4935923414e-01
	7	2.1165351693e-01	5.4440142107e-01
	8	-6.7366756980e-02	3.2169170065e-01
	9	-9.0159055096e-03	-2.0365603789e-02
	10	2.3049919460e-02	-5.2066376096e-02
k=2	0	6.3813937293e-02	1.4204323343e-01
	1	-6.2433761707e-01	-5.3003493793e-01
	2	-7.3965386969e-03	1.7695538529e-02
	4	-6.2421804274e-01	-4.0650580195e-01
	5	-1.3488453082e-02	-4.9029535249e-01
	6	4.3699790930e-01	-9.0099784088e-02
	7	1.4222943082e-01	4.3727314163e-01
	8	-6.9374369289e-02	3.0298274581e-01
	9	-5.5998777298e-03	-1.5509344332e-02
	10	1.4316557613e-02	-3.9650941036e-02

Note: For the right edge ($x = 1$) the coefficients $\{H^1_{0,n}\}$ are related to the scaling function $\varphi^1_{j,2^j-3}$ and are listed from right to left. The case n=3 corresponds to the scaling function we have removed and will therefore not appear.

Table 4: The left and right wavelet filter coefficients, $G^0_{k,n}$ and $G^1_{k,n}$, for the construction with Neumann homogeneous boundary conditions $f^{(1)}(0) = f^{(1)}(1) = 0$.

	n	$G^0_{k,n}$	$G^1_{k,n}$
k=0	0	2.1578281914e-01	2.3494978252e-01
	1	4.9572192648e-01	4.8386498781e-01
	2	7.2238470640e-01	5.2859034242e-01
	4	-3.9185512760e-01	-6.4278506397e-01
	5	-1.2862537931e-01	4.9429597015e-02
	6	9.6457537590e-02	1.2296776567e-01
	7	6.4796296855e-02	-1.9638518548e-02
	8	-4.7138054627e-02	-1.1782477983e-02
	9	-2.2555618915e-03	7.3127496136e-04
	10	5.7665333650e-03	1.8695658406e-03
k=1	0	5.2517520677e-13	5.5828490449e-14
	1	2.4258207630e-02	2.0458047986e-01
	2	6.0376140438e-03	5.9531516311e-01
	4	2.2848095434e-01	5.4090218399e-01
	5	-5.1810004520e-01	-4.1031183125e-01
	6	5.6358588265e-01	-3.4180312799e-01
	7	-5.2660919644e-01	1.2529183073e-01
	8	2.8410901284e-01	1.0078064242e-01
	9	2.0215169170e-02	-4.1780815641e-03
	10	-5.1681777360e-02	-1.0681616333e-02
k=2	0	1.9660656197e-12	-2.7202352286e-12
	1	3.0437460103e-12	-9.0382597736e-12
	2	5.5424594564e-02	1.5213229661e-02
	4	-2.7483194254e-01	1.2607094728e-01
	5	6.0853476157e-01	-5.5744040614e-01
	6	-1.0850026370e-01	7.6287424658e-01
	7	-6.0447233132e-01	-2.8671529090e-01
	8	4.1282047408e-01	-8.8419425836e-02
	9	2.1554056091e-02	1.2267369323e-02
	10	-5.5104754193e-02	3.1362559711e-02
k=3	0	1.9941691330e-01	1.8775994937e-01
	1	5.3813012840e-01	6.1793471320e-01
	2	-6.8862102186e-01	-6.0431561252e-01
	4	-4.2066297006e-01	-3.7019350637e-02
	5	-7.4122879021e-02	-3.8404010349e-01
	6	1.1518683604e-01	-2.0032106281e-01
	7	2.1861286109e-02	1.3094817230e-01
	8	-1.5750923988e-02	1.0691409643e-01
	9	-7.6389913001e-04	-4.3365631441e-03
	10	1.9529722670e-03	-1.1086787800e-02

Note: For the right edge ($x = 1$) the coefficients $\{G^1_{0,n}\}$ are related to the wavelet $\psi^1_{j,2^j-1}$ and are listed from right to left. The case n=3 corresponds to the scaling functions we have removed and will therefore not appear.

Table 5: **The first four moments for the 0 edge scaling function of Figure 2 (i.e satisfying Dirichlet homogeneous boundary conditions).**

| X_l	$\varphi^0_{0,0}$	$\varphi^0_{0,1}$	$\varphi^0_{0,2}$
l=0	-1.80518411015e+00	1.58610287947e-01	1.42577315923e-01
l=1	-3.45978990889e+00	-1.34279194417e+00	-1.66298580543e-13
l=2	-8.12722032481e+00	-5.41163047176e+00	7.89305924956e-01
l=3	-2.13375831107e+01	-1.76739256352e+01	4.70776245989e+00

Note: For this case the monomial x could be expanded as a linear combination of $\varphi^0_{0,0}$ and $\varphi^0_{0,1}$. This explains the zero value of the second moment of $\varphi^0_{0,2}$.

Table 6: **The first four moments for the 1 edge scaling function of Figure 1 (i.e satisfying Dirichlet homogeneous boundary conditions).**

X_l	$\varphi1_{0,0}$	$\varphi^1_{0,1}$	$\varphi^1_{0,2}$
l=0	-2.03632572831e+00	2.11233673136e-01	2.34792863276e-01
l=1	-5.17867977397e+00	-1.87776044074e+00	-4.21625644274e-13
l=2	-1.57657032623e+01	-1.04701396775e+01	1.27839462595e+00
l=3	-5.24899484894e+01	-4.53961860524e+01	1.15515969500e+01

Note: same remarks as Table 5 for the value of the second moment of $\varphi^1_{0,2}$.

a)The three scaling functions for the left edge
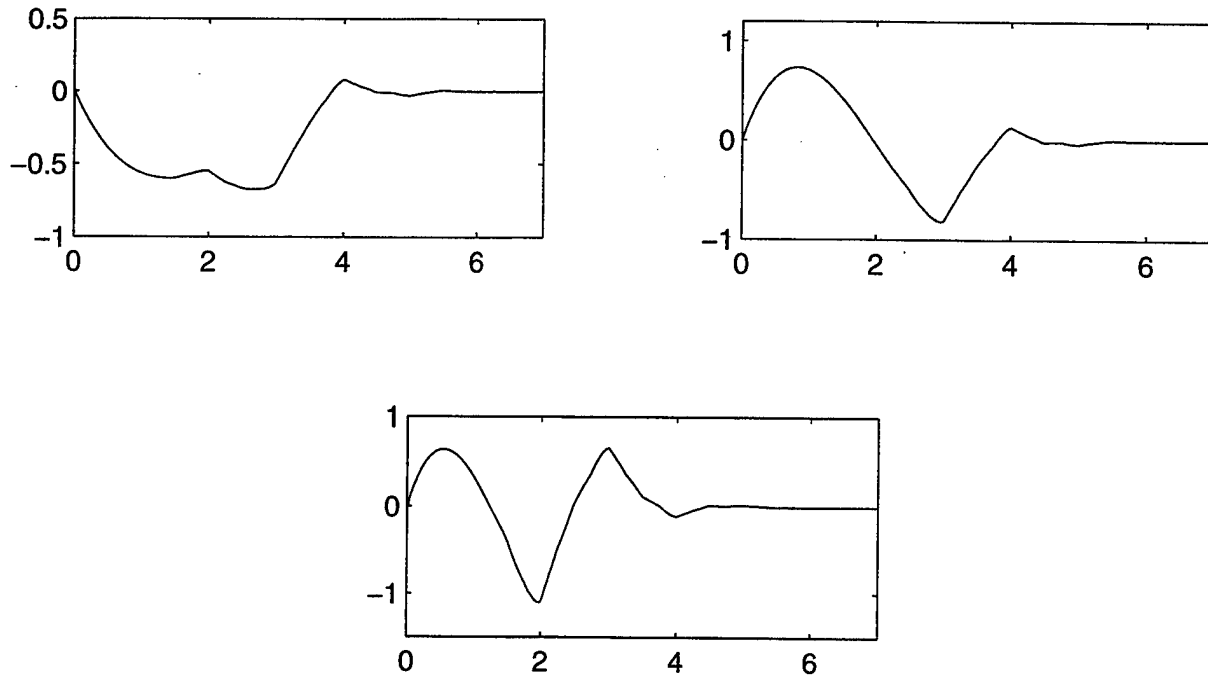
b)The three scaling functions for the right edge
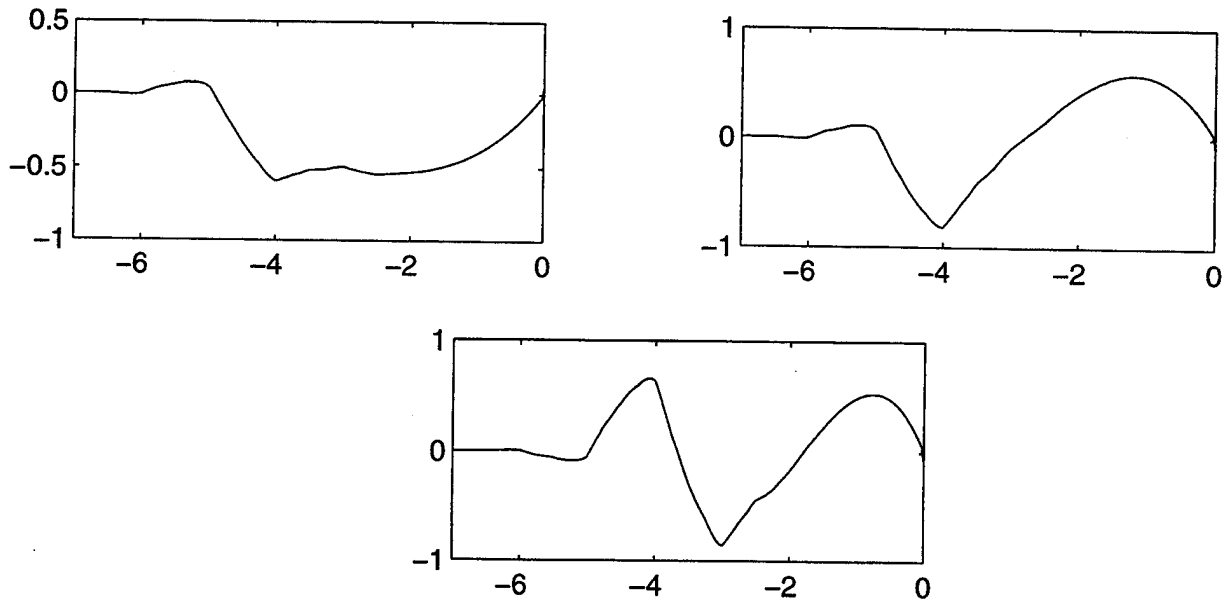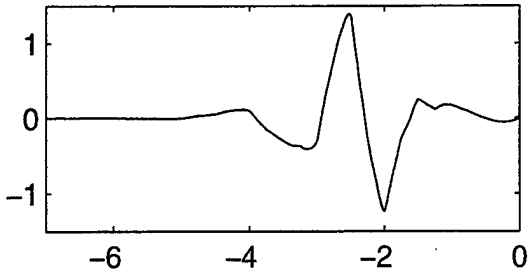

FIG. 2. *The six edge scaling functions for the case N=4, represented at scale j=0 (i.e. on [0,+∞[for the left edge and]−∞,0] for the right edge). All these scaling functions satisfy Dirichlet homogeneous boundary conditions at 0.*
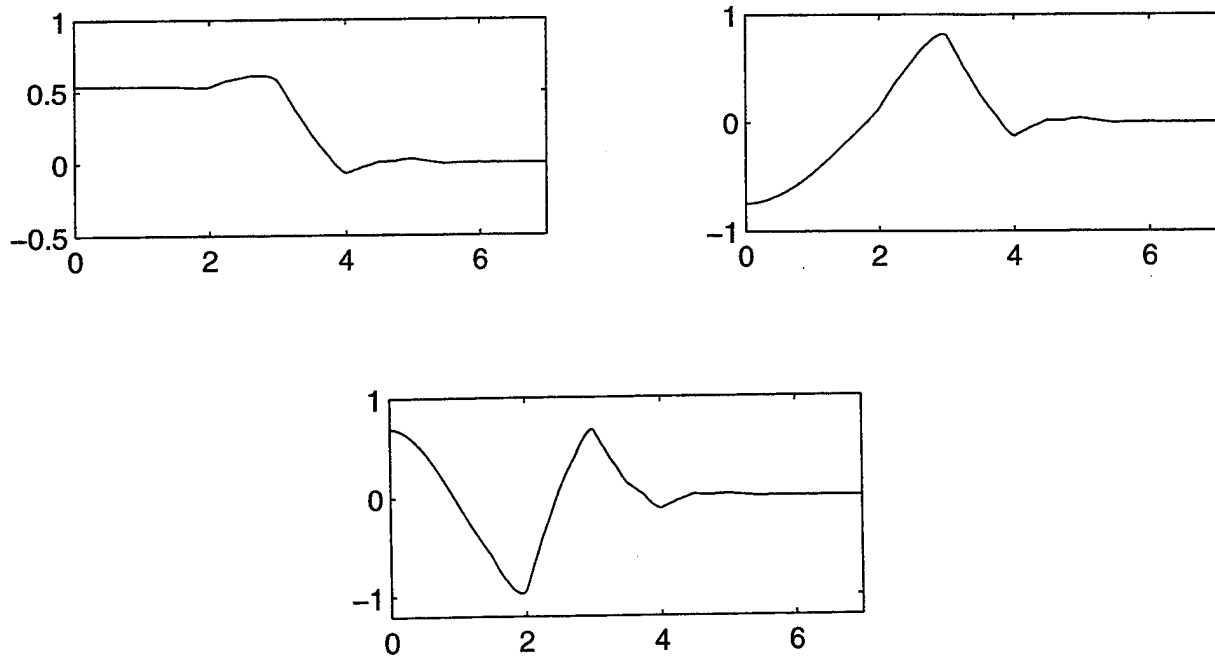
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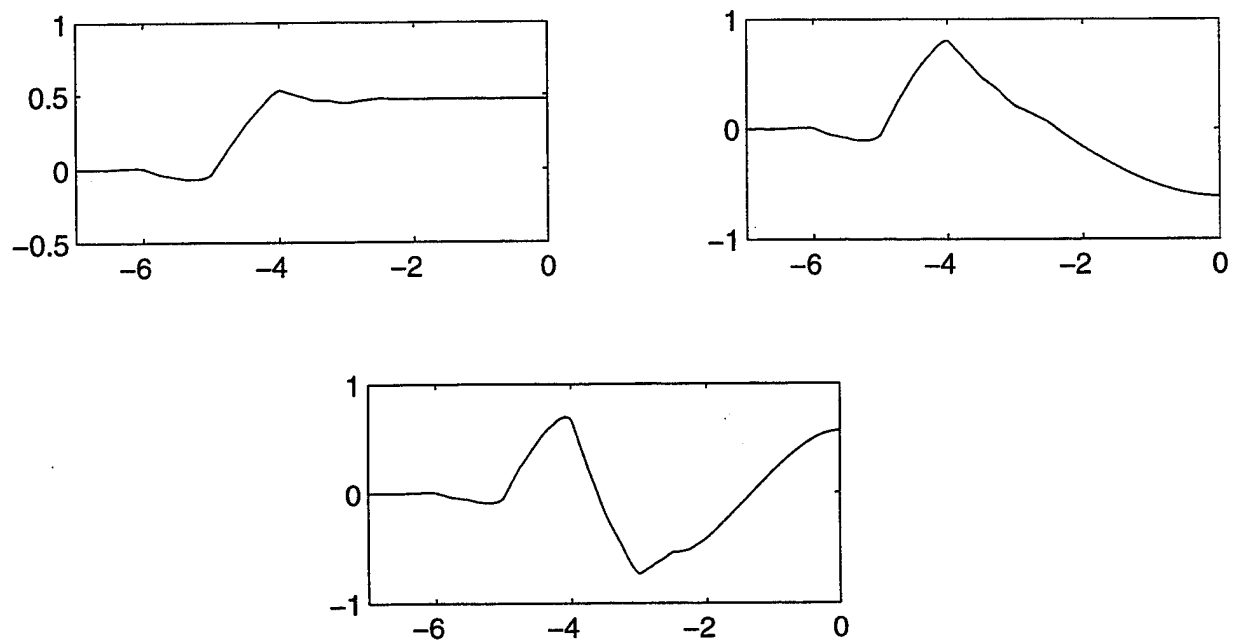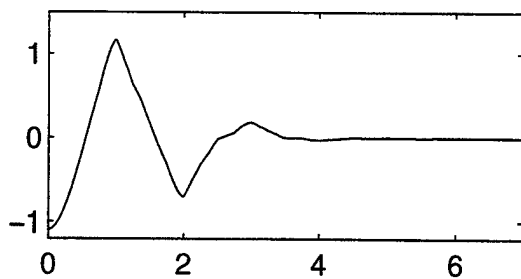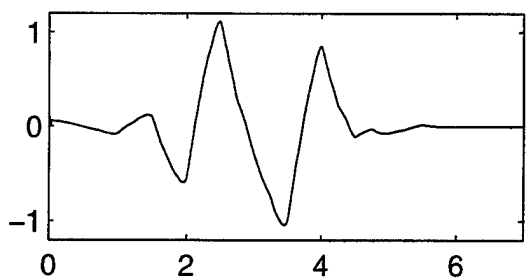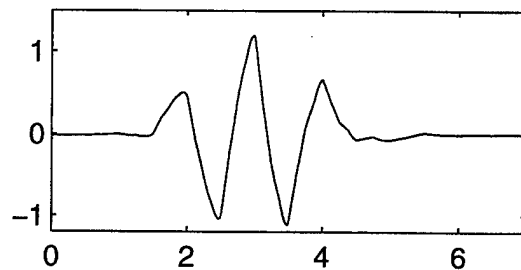a)The four wavelets for the left edge
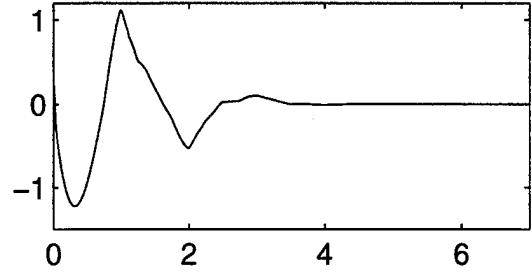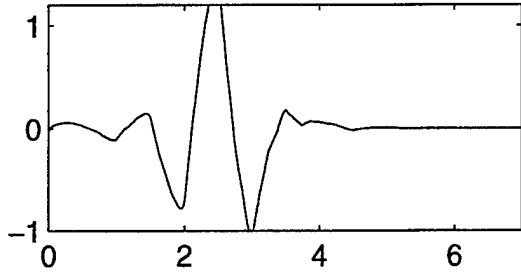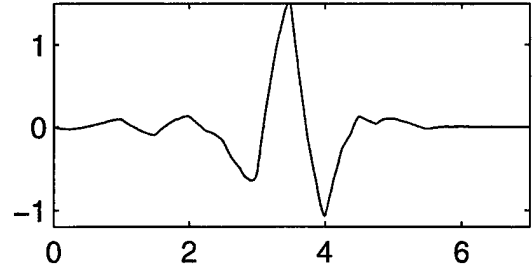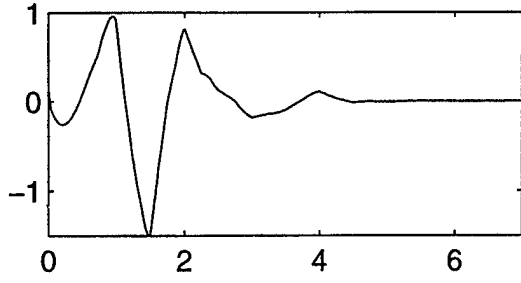
b)The four wavelets for the right edge
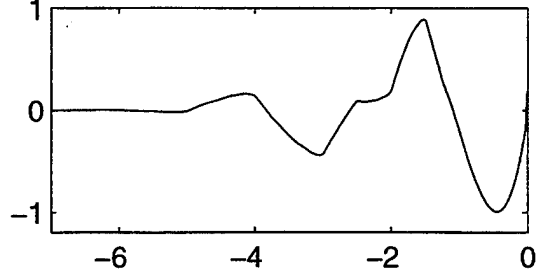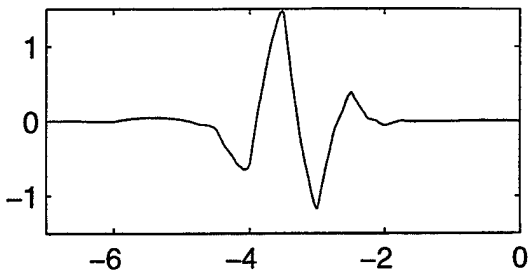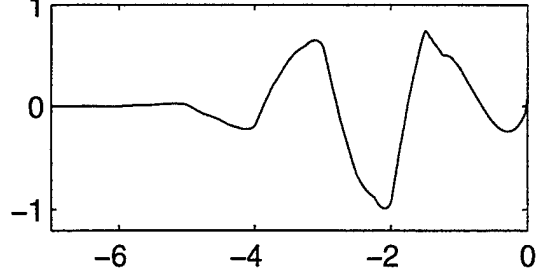
FIG. 3. *The eight edge wavelets for the case N=4, represented at scale j=0 (i.e on [0,+∞[for the left edge and] −∞,0] for the right edge). All these wavelets satisfy Dirichlet homogeneous boundary conditions at 0.*

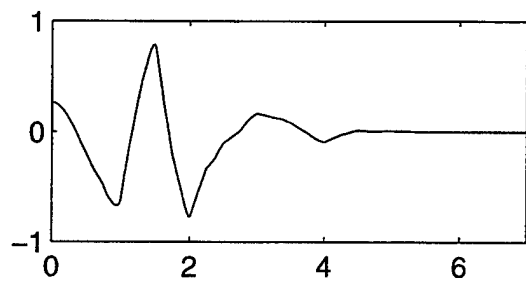a)The three scaling functions for the left edge
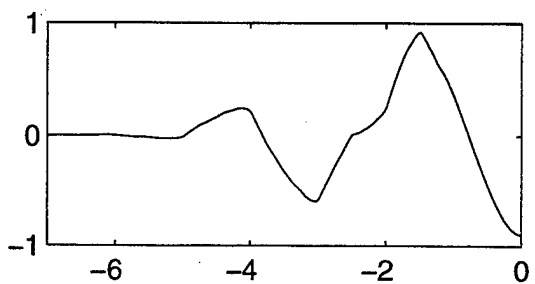

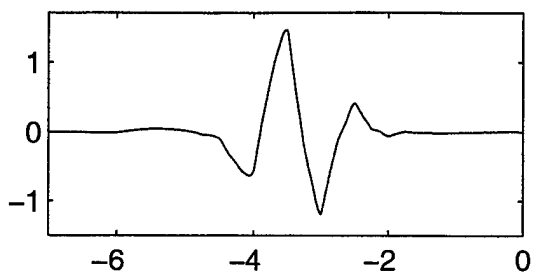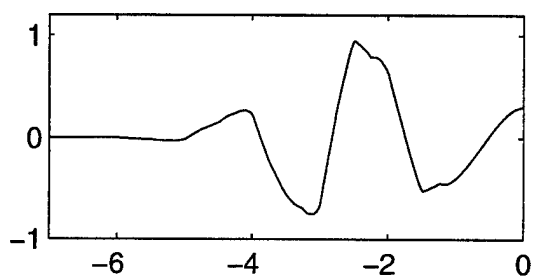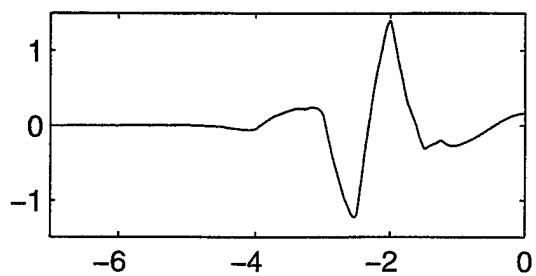b)The three scaling functions for the right edge

FIG. 4. *Same as in Fig. 2 for the Neumann homogeneous boundary conditions*

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a)The four wavelets for the left edge

b)The four wavelets for the right edge

FIG. 5. *Same as in Fig. 3 for the Neumann homogeneous boundary conditions*

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G. Chiavassa
J. Liandrat

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Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23681-0001

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We construct compactly supported wavelet bases satisfying homogeneous boundary conditions on the interval $[0,1]$. The maximum features of multiresolution analysis on the line are retained, including polynomial approximation and tree algorithms. The case of $H_0^1([0,1])$ is detailed, and numerical values, required for the implementation, are provided for the Neumann and Dirichlet boundary conditions.

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